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JAN 81 R E BECHHOFER, A C TAMHANE DAAG29-80-C-0036
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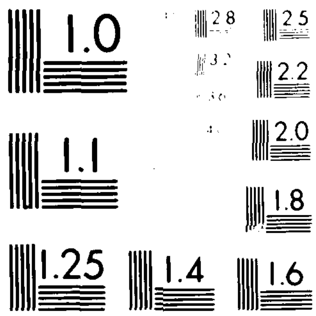

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TABLES OF OPTIMAL ALLOCATIONS OF OBSERVATIONS FOR COMPARING TREATMENTS WITH A CONTROL

by

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ABSTRACT

The problem under consideration is that of estimating simultaneously the differences between the means of $p \geq 2$ test treatments and the mean of a control treatment when the population variances of all $p + 1$ treatments are known. Tables are given which permit the experimenter to find the minimal total number of experimental units, and the optimal allocation of these units among the $p + 1$ treatments, in order to make one-sided or two-sided confidence interval estimates of the differences of interest. These intervals achieve a specified joint confidence coefficient $1-\alpha$ for a specified allowance or "yardstick" associated with the common width of the intervals. The computations for these tables are based on the results of Bechhofer (1969) for one-sided comparisons, and Bechhofer and Nocturne (1972) for two-sided comparisons.

Key Words and Phrases: Multiple comparisons with a control, Dunnett's procedure, optimal allocation of observations, one-sided comparisons, two-sided comparisons, joint confidence coefficient, completely randomized design.

1. INTRODUCTION

The problem of comparing simultaneously $p \geq 2$ test treatments with a control treatment arises frequently in applied research. Dunnett (1955), (1964) considered this problem and provided constants necessary to make joint $100(1-\alpha)$ percent confidence statements (either one-sided or two sided) between the mean of each of the test treatments and the mean of the control treatment when the common variance of the $p + 1$ treatments is unknown.

An important design decision in this problem is that of how to allocate the experimental units optimally among the test treatments and the control treatment when the $p + 1$ variances are known and possibly unequal. Bechhofer (1969) (hereinafter referred to as B1) gave a solution to this problem for <u>one-sided</u> comparisons; this solution is globally optimal if the variances of the p test treatments are equal, and optimal in a restricted sense if these variances are unequal. Bechhofer and Nocturne (1972) (hereinafter referred to as B2) generalized these results to <u>two-sided</u> comparisons. Bechhofer and Turnbull (1971) gave a globally optimal solution to this problem for <u>one-sided</u> comparisons when the p <u>test</u> variances are known and unequal.

Only small illustrative sets of tables of optimal allocations (all for $p = 2$) were given in B1 and B2. In the present paper we give an extensive set of tables for $p = 2(1)10$ both for joint one-sided or joint two-sided comparisons based on the formulae given in B1 and B2. (See Remark 2.2 for the case $p = 1$.) For such comparisons these tables can be used to determine the smallest total number of observations necessary to guarantee selected confidence coefficients (0.75(0.05)0.95, 0.99) for given specified allowance or "yardstick" associated with the common "width" of the confidence intervals; the tables also tell how to allocate these observations optimally to the $p + 1$ treatments.

Remark 1.1: The present paper (and all of the aforementioned papers) deals with the case in which a completely randomized design is to be employed. However, many practical situations may require the blocking of experimental units. If the block size is large enough to accommodate one replication of all of the test treatments and additional control treatments as well, then the optimal allocations in the present paper can be used. If the blocks have a common size $k < p + 1$, i.e., if the $p + 1$ treatments are to be compared in incomplete blocks, then entirely new considerations are required to determine the optimal incomplete block design. This problem is considered in Bechhofer and Tamhane (1981).

In Section 2 we introduce our notation and pose the optimal allocation problem both for one-sided and two-sided comparisons. The tables along with an explanation of how they are to be used are given in Sections 3 and 4. Section 5 quantifies the loss incurred if equal allocation is used instead of the optimal allocation. The formulae used in the computation of Tables 1 through 9, and details of the computations are given in the Appendix.

2. NOTATION AND STATEMENT OF THE OPTIMAL ALLOCATION PROBLEMS

Let the treatments be indexed by $0,1,\ldots,p$ with 0 denoting the control treatment and $1,2,\ldots,p$ denoting the $p \geq 2$ test treatments. We assume that the observations X_{ij} $(j = 1,2,\ldots)$ on the ith treatment are normally distributed with unknown mean μ_i and known variance σ_i^2 $(0 \leq i \leq p)$, and that all observations are mutually independent. Based on $N_i \geq 1$ observations on the ith treatment $(0 \leq i \leq p)$ it is desired to make either

(I) A $100(1-\alpha)$ percent joint one-sided confidence statement of the form

$$\{\mu_0 - \mu_i \leq \bar{x}_0 - \bar{x}_i + d \ (1 \leq i \leq p)\}, \tag{2.1}$$

or

(II) A $100(1-\alpha)$ percent joint two-sided confidence statement of the form

$$\{\bar{x}_0 - \bar{x}_i - d \leqq \mu_0 - \mu_i \leqq \bar{x}_0 - \bar{x}_i + d \quad (1 \leqq i \leqq p)\}. \tag{2.2}$$

In (2.1) and (2.2), $\bar{x}_i$ is the observed value of the random variable $\bar{X}_i = \sum_{j=1}^{N_i} X_{ij}/N_i$ $(0 \leqq i \leqq p)$, and $d > 0$ is a specified allowance.

The optimal allocation problem is that of finding the allocation vector $(N_0, N_1, \ldots, N_p)$ which for known $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$ and specified $1-\alpha$ and d, minimizes the total sample size $N = \sum_{i=0}^{p} N_i$ subject to

$$P\{\mu_0 - \mu_i \leqq \bar{X}_0 - \bar{X}_i + d \quad (1 \leqq i \leqq p)\} \geqq 1 - \alpha \tag{2.3}$$

for one-sided comparisons, and

$$P\{\bar{X}_0 - \bar{X}_i - d \leqq \mu_0 - \mu_i \leqq \bar{X}_0 - \bar{X}_i + d \quad (1 \leqq i \leqq p)\} \geqq 1 - \alpha \tag{2.4}$$

for two-sided comparisons. For both cases we denote the optimal allocation by $(\hat{N}_0, \hat{N}_1, \ldots, \hat{N}_p)$ and the smallest total sample size by $\hat{N} = \sum_{i=0}^{p} \hat{N}_i$ (the particular case under consideration being clear from the context).

Remark 2.1: This same optimal allocation $(\hat{N}_0, \hat{N}_1, \ldots, \hat{N}_p)$ **maximizes** the joint confidence coefficient for known $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$ and specified total sample size $N = \sum_{i=0}^{p} \hat{N}_i$ and d.

Continuous approximations to the probabilities (2.3) and (2.4) are obtained in B1 and B2, respectively, by letting

$$\gamma_i = N_i / \sum_{i=0}^{p} N_i \quad (0 \leqq i \leqq p), \tag{2.5}$$

and regarding the γ_i as nonnegative continuous variables satisfying $\sum_{i=0}^{p} \gamma_i = 1$. The solutions given in B1 and B2 give optimal allocations

for one-sided and two-sided comparisons under the restriction that the standard errors $\sigma_i/\sqrt{N_i}$ $(1 \leq i \leq p)$ of the test treatment means are equal, i.e., $\sigma_i^2/\gamma_i = \sigma_j^2/\gamma_j$ $(i \neq j; 1 \leq i,j \leq p)$; if $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_p^2$, then the solutions give globally optimal allocations.

Under the stated restriction and using the continuous approximation, the probabilities (2.3) and (2.4) can be shown for given p to depend on $(\sigma_0^2, \sigma_1^2, \ldots, \sigma_p^2)$, N and d only through γ_0,

$$\lambda = d\sqrt{N}/\sigma_0 \tag{2.6}$$

and

$$\beta = \left(\sum_{i=1}^{p} \sigma_i^2\right)/\sigma_0^2. \tag{2.7}$$

For given p and β, and specified $1-\alpha$ the optimal solutions which we denote by $(\hat{\gamma}_0, \hat{\lambda})$ are uniquely determined. The simultaneous equations which yield these solutions are given in the Appendix.

Remark 2.2: It should be noted that for $p = 1$ the globally optimal allocation for one-sided and for two-sided comparisons is $\sigma_0/\hat{N}_0 = \sigma_1/\hat{N}_1$. Then $\hat{N} = \{(\sigma_0 + \sigma_1)z_\alpha/d\}^2$ and $\hat{N}_i = \hat{N}\sigma_i/(\sigma_0 + \sigma_1)$ $(i = 0,1)$ for one-sided comparisons, the same expressions holding for two-sided comparisons with $z_{\alpha/2}$ replacing z_α; here z_α is the upper α-point of the standard normal distribution.

3. DESCRIPTION OF THE TABLES

Tables 1 through 9 give values of $(\hat{\gamma}_0, \hat{\lambda})$ both for one-sided and two-sided comparisons for $p = 2(1)10$, respectively. The tabulated values of $\hat{\gamma}_0$ are correct to within one in the fourth decimal place while the tabulated values of $\hat{\lambda}$ are rounded up in the fourth decimal place to guarantee a joint

confidence coefficient $\geq 1-\alpha$ for the tabulated value of $\hat{\gamma}_0$. For each value of p the tabulations are made for $1-\alpha = 0.75(0.05)0.95, 0.99$ and $\beta = p/2, p, 3p/2$ and $2p$. From (2.7) we see that the tables can be used for the special case $\sigma_1^2 = \ldots = \sigma_p^2 = \sigma^2$ (say) (in which case the allocations are globally optimal) when $\sigma^2 = c\sigma_0^2$ for $c = 1/2, 1, 3/2, 2$. In particular, the $\beta = p$ column can be used for the special case $\sigma_0^2 = \sigma_1^2 = \ldots = \sigma_p^2$.

An examination of the tables shows that for fixed p, β and $1-\alpha$, we have $\hat{\gamma}_0$ and $\hat{\lambda}$ in the two-sided case always greater than the corresponding $\hat{\gamma}_0$ and $\hat{\lambda}$, respectively, in the one-sided case. In both cases for fixed p and β we have $\hat{\gamma}_0$ increasing with $1-\alpha$ and approaching the limit $1/(1 + \sqrt{\beta})$ as $1-\alpha$ approaches unity (and hence $\hat{\gamma}_0/\hat{\gamma}_i \to \sqrt{\beta}\ \sigma_0^2/\sigma_i^2$ for $1 \leq i \leq p$). This limiting result has been proved analytically in B1 and B2 for the one-sided and two-sided cases, respectively. For $\sigma_0^2 = \sigma_1^2 = \ldots = \sigma_p^2$ this gives the limiting result that $\hat{\gamma}_0/\hat{\gamma}_i \to \sqrt{p}$ $(1 \leq i \leq p)$ which leads to the recommendation of Dunnett (1955), pp. 1106-1107 and (1964), pp. 486-487.

4. USE OF THE TABLES

The tables of $(\hat{\gamma}_0, \hat{\lambda})$ are to be used as follows: p and σ_i^2 $(0 \leq i \leq p)$ are given as data of the problem; these determine β. The experimenter specifies d, i.e., his allowance, and his one-sided or two-sided joint confidence coefficient $1-\alpha$. Then p, β, and his one-sided or two-sided $1-\alpha$ determine $(\hat{\gamma}_0, \hat{\lambda})$. The smallest total sample size $\hat{N}$ is then the smallest integer $\geq (\hat{\lambda}\ \sigma_0/d)^2$. The optimal allocations are given by $\hat{N}_0 = \hat{\gamma}_0\hat{N}$ (to the nearest integer) and $\hat{N}_i = (\hat{N}-\hat{N}_0)\sigma_i^2/\beta\sigma_0^2$ (to the nearest integer) for $(1 \leq i \leq p)$; these <u>approximate</u> <u>integer</u> allocations which were obtained by using the continuous approximations will be very close to the <u>exact</u> <u>integer</u> allocations if $\hat{N}$ is large.

Table 1

Optimal allocation [1] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient 1-α

$p = 2$

1-α		β			
		p/2	p	3p/2	2p
0.75	1-sided	0.4245 2.0074	0.3519 2.4818	0.3115 2.8420	0.2841 3.1441
	2-sided	0.4638 2.9007	0.3868 3.5403	0.3435 4.0266	0.3141 4.4348
0.80	1-sided	0.4417 2.3226	0.3666 2.8556	0.3246 3.2608	0.2961 3.6009
	2-sided	0.4691 3.1474	0.3908 3.8352	0.3468 4.3588	0.3169 4.7986
0.85	1-sided	0.4559 2.6885	0.3788 3.2904	0.3354 3.7486	0.3060 4.1334
	2-sided	0.4745 3.4441	0.3950 4.1897	0.3503 4.7581	0.3198 5.2357
0.90	1-sided	0.4683 3.1481	0.3893 3.8376	0.3448 4.3631	0.3146 4.8048
	2-sided	0.4802 3.8298	0.3995 4.6506	0.3540 5.2771	0.3231 5.8039
0.95	1-sided	0.4801 3.8298	0.3992 4.6510	0.3536 5.2780	0.3225 5.8053
	2-sided	0.4866 4.4228	0.4046 5.3600	0.3583 6.0763	0.3268 6.6791
0.99	1-sided	0.4916 5.1150	0.4085 6.1894	0.3616 7.0114	0.3296 7.7036
	2-sided	0.4939 5.5882	0.4103 6.7571	0.3630 7.6520	0.3309 8.4058
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.5000	0.4142	0.3660	0.3333

[1] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 2

Optimal allocation[1/] on the control ($\hat{\gamma}_0$)
and associated $\hat{\lambda}$ to achieve a given joint
confidence coefficient $1-\alpha$

$p = 3$

$1-\alpha$		β: p/2	p	3p/2	2p
0.75	1-sided	0.3567 2.6508	0.2920 3.3507	0.2568 3.8812	0.2333 4.3271
	2-sided	0.4032 3.6100	0.3322 4.4951	0.2931 5.1675	0.2669 5.7314
0.80	1-sided	0.3766 2.9904	0.3086 3.7561	0.2713 4.3378	0.2464 4.8259
	2-sided	0.4099 3.8746	0.3371 4.8148	0.2971 5.5299	0.2702 6.1301
0.85	1-sided	0.3936 3.3832	0.3227 4.2266	0.2837 4.8682	0.2576 5.4067
	2-sided	0.4168 4.1925	0.3423 5.1989	0.3013 5.9654	0.2738 6.6092
0.90	1-sided	0.4089 3.8757	0.3353 4.8186	0.2948 5.5366	0.2675 6.1398
	2-sided	0.4241 4.6057	0.3479 5.6986	0.3059 6.5320	0.2777 7.2326
0.95	1-sided	0.4240 4.6059	0.3475 5.6993	0.3053 6.5336	0.2770 7.2350
	2-sided	0.4324 5.2417	0.3543 6.4690	0.3112 7.4064	0.2822 8.1950
0.99	1-sided	0.4389 5.9855	0.3591 7.3725	0.3151 8.4332	0.2856 9.3263
	2-sided	0.4420 6.4953	0.3613 7.9931	0.3169 9.1395	0.2871 10.1049
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.4495	0.3660	0.3204	0.2899

[1/]The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 3

Optimal allocation[1] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient 1-α

p = 4

1-α		β			
		p/2	p	3p/2	2p
0.75	1-sided	0.3163 3.1961	0.2570 4.0932	0.2251 4.7742	0.2040 5.3454
	2-sided	0.3645 4.2095	0.2978 5.3084	0.2617 6.1432	0.2376 6.8435
0.80	1-sided	0.3366 3.5555	0.2735 4.5245	0.2395 5.2607	0.2169 5.8785
	2-sided	0.3716 4.4888	0.3031 5.6486	0.2659 6.5307	0.2411 7.2711
0.85	1-sided	0.3543 3.9706	0.2879 5.0246	0.2521 5.8263	0.2281 6.4994
	2-sided	0.3791 4.8244	0.3085 6.0574	0.2703 6.9963	0.2448 7.7850
0.90	1-sided	0.3706 4.4904	0.3011 5.6535	0.2634 6.5394	0.2383 7.2836
	2-sided	0.3870 5.2607	0.3144 6.5894	0.2750 7.6026	0.2488 8.4553
0.95	1-sided	0.3868 5.2609	0.3139 6.5904	0.2743 7.6047	0.2480 8.4575
	2-sided	0.3960 5.9326	0.3211 7.4107	0.2805 8.5396	0.2535 9.4894
0.99	1-sided	0.4031 6.7197	0.3263 8.3762	0.2846 9.6432	0.2569 10.7100
	2-sided	0.4064 7.2602	0.3286 9.0412	0.2865 10.4043	0.2585 11.5523
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.4142	0.3333	0.2899	0.2612

[1] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 4

Optimal allocation [1/] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient $1-\alpha$

$p = 5$

$1-\alpha$		β			
		$p/2$	p	$3p/2$	$2p$
0.75	1-sided	0.2884 3.6806	0.2331 4.7562	0.2036 5.5731	0.1842 6.2583
	2-sided	0.3367 4.7403	0.2735 6.0322	0.2395 7.0138	0.2170 7.8374
0.80	1-sided	0.3086 4.0569	0.2493 5.2096	0.2176 6.0857	0.1967 6.8210
	2-sided	0.3440 5.0325	0.2787 6.3901	0.2437 7.4229	0.2205 8.2830
0.85	1-sided	0.3263 4.4911	0.2635 5.7351	0.2299 6.6817	0.2076 7.4764
	2-sided	0.3516 5.3835	0.2842 6.8203	0.2481 7.9148	0.2241 8.8342
0.90	1-sided	0.3428 5.0344	0.2766 6.3961	0.2411 7.4334	0.2176 8.3050
	2-sided	0.3597 5.8397	0.2901 7.3805	0.2528 8.5556	0.2281 9.5434
0.95	1-sided	0.3594 5.8401	0.2896 7.3819	0.2521 8.5582	0.2273 9.5475
	2-sided	0.3690 6.5430	0.2969 8.2463	0.2583 9.5473	0.2327 10.6420
0.99	1-sided	0.3762 7.3681	0.3021 9.2662	0.2624 10.7182	0.2361 11.9406
	2-sided	0.3796 7.9354	0.3044 9.9699	0.2642 11.5272	0.2377 12.8389
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3874	0.3090	0.2675	0.2403

[1/] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 5

Optimal allocation 1/ on the control ($\hat{\gamma}_0$)
and associated $\hat{\lambda}$ to achieve a given joint
confidence coefficient 1-α

p = 6

1-α		β p/2	β p	β 3p/2	β 2p
0.75	1-sided	0.2677 4.1224	0.2154 5.3627	0.1878 6.3052	0.1696 7.0960
0.75	2-sided	0.3154 5.2230	0.2549 6.6925	0.2227 7.8095	0.2014 8.7469
0.80	1-sided	0.2874 4.5136	0.2311 5.8357	0.2013 6.8410	0.1816 7.6848
0.80	2-sided	0.3227 5.5266	0.2601 7.0663	0.2268 8.2380	0.2048 9.2218
0.85	1-sided	0.3050 4.9646	0.2451 6.3838	0.2132 7.4639	0.1922 8.3710
0.85	2-sided	0.3303 5.8913	0.2655 7.5156	0.2311 8.7532	0.2084 9.7930
0.90	1-sided	0.3215 5.5289	0.2580 7.0734	0.2242 8.2502	0.2020 9.2391
0.90	2-sided	0.3384 6.3655	0.2714 8.1010	0.2358 9.4249	0.2123 10.5380
0.95	1-sided	0.3381 6.3659	0.2708 8.1027	0.2350 9.4282	0.2114 10.5429
0.95	2-sided	0.3477 7.0968	0.2781 9.0067	0.2411 10.4658	0.2168 11.6936
0.99	1-sided	0.3551 7.9560	0.2832 10.0755	0.2451 11.6970	0.2201 13.0624
0.99	2-sided	0.3584 8.5476	0.2855 10.8140	0.2469 12.5493	0.2216 14.0108
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3660	0.2899	0.2500	0.2240

1/ The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 6

Optimal allocation[1/] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient 1-α

p = 7

1-α		β p/2	β p	β 3p/2	β 2p
0.75	1-sided	0.2513 4.5320	0.2016 5.9265	0.1754 6.9865	0.1583 7.8761
	2-sided	0.2983 5.6693	0.2402 7.3047	0.2094 8.5482	0.1891 9.5920
0.80	1-sided	0.2706 4.9366	0.2168 6.4170	0.1885 7.5431	0.1698 8.4886
	2-sided	0.3055 5.9833	0.2452 7.6929	0.2134 8.9942	0.1924 10.0871
0.85	1-sided	0.2880 5.4027	0.2304 6.9855	0.2000 8.1905	0.1801 9.2026
	2-sided	0.3130 6.3604	0.2506 8.1597	0.2176 9.5308	0.1959 10.6830
0.90	1-sided	0.3042 5.9860	0.2431 7.7010	0.2107 9.0081	0.1895 10.1056
	2-sided	0.3211 6.8509	0.2563 8.7680	0.2221 10.2307	0.1997 11.4607
0.95	1-sided	0.3208 6.8515	0.2557 8.7700	0.2213 10.2345	0.1988 11.4663
	2-sided	0.3304 7.6079	0.2629 9.7101	0.2273 11.3165	0.2040 12.6683
0.99	1-sided	0.3377 8.4982	0.2679 10.8236	0.2312 12.6029	0.2072 14.1013
	2-sided	0.3410 9.1120	0.2701 11.5941	0.2329 13.4947	0.2086 15.0956
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3483	0.2743	0.2358	0.2109

[1/] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 7

Optimal allocation[1] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient 1-α

p = 8

1-α		β: p/2	p	3p/2	2p
0.75	1-sided	0.2380 4.9161	0.1904 6.4561	0.1654 7.6273	0.1491 8.6103
	2-sided	0.2841 6.0870	0.2280 7.8787	0.1984 9.2416	0.1790 10.3857
0.80	1-sided	0.2569 5.3328	0.2052 6.9627	0.1780 8.2030	0.1603 9.2444
	2-sided	0.2912 6.4105	0.2330 8.2802	0.2023 9.7038	0.1822 10.9000
0.85	1-sided	0.2739 5.8129	0.2184 7.5499	0.1893 8.8727	0.1702 9.9840
	2-sided	0.2987 6.7991	0.2382 8.7630	0.2064 10.2601	0.1856 11.5182
0.90	1-sided	0.2899 6.4135	0.2308 8.2892	0.1997 9.7192	0.1793 10.9211
	2-sided	0.3066 7.3046	0.2438 9.3926	0.2108 10.9860	0.1892 12.3260
0.95	1-sided	0.3063 7.3053	0.2432 9.3949	0.2100 10.9904	0.1883 12.3325
	2-sided	0.3158 8.0853	0.2502 10.3684	0.2158 12.1133	0.1934 13.5818
0.99	1-sided	0.3230 9.0045	0.2551 11.5233	0.2196 13.4509	0.1965 15.0743
	2-sided	0.3263 9.6389	0.2573 12.3235	0.2213 14.3794	0.1979 16.1112
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3333	0.2612	0.2240	0.2000

[1] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 8

Optimal allocation[1/] on the control $(\hat{\gamma}_0)$ and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient $1-\alpha$

$p = 9$

$1-\alpha$		β			
		$p/2$	p	$3p/2$	$2p$
0.75	1-sided	0.2269 5.2794	0.1811 6.9578	0.1572 8.2346	0.1415 9.3066
	2-sided	0.2721 6.4812	0.2177 8.4214	0.1892 9.8977	0.1705 11.1372
0.80	1-sided	0.2453 5.7073	0.1954 7.4792	0.1693 8.8280	0.1523 9.9608
	2-sided	0.2791 6.8135	0.2226 8.8352	0.1930 10.3750	0.1736 11.6685
0.85	1-sided	0.2620 6.2002	0.2083 8.0837	0.1802 9.5186	0.1619 10.7242
	2-sided	0.2864 7.2128	0.2277 9.3331	0.1969 10.9496	0.1769 12.3084
0.90	1-sided	0.2777 6.8170	0.2204 8.8452	0.1903 10.3919	0.1708 11.6921
	2-sided	0.2942 7.7324	0.2331 9.9824	0.2012 11.7000	0.1804 13.1444
0.95	1-sided	0.2939 7.7331	0.2325 9.9851	0.2004 11.7048	0.1795 13.1515
	2-sided	0.3033 8.5352	0.2394 10.9897	0.2063 12.8660	0.1845 14.4452
0.99	1-sided	0.3104 9.4814	0.2441 12.1834	0.2097 14.2515	0.1874 15.9932
	2-sided	0.3136 10.1351	0.2463 13.0113	0.2113 15.2142	0.1887 17.0699
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3204	0.2500	0.2139	0.1907

[1/] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

Table 9

Optimal allocation[1/] on the control ($\hat{\gamma}_0$) and associated $\hat{\lambda}$ to achieve a given joint confidence coefficient 1-α

p = 10

1-α		β			
		p/2	p	3p/2	2p
0.75	1-sided	0.2174 5.6252	0.1731 7.4359	0.1501 8.8138	0.1351 9.9708
	2-sided	0.2618 6.8558	0.2089 8.9378	0.1813 10.5225	0.1632 11.8532
0.80	1-sided	0.2354 6.0635	0.1871 7.9712	0.1619 9.4237	0.1455 10.6438
	2-sided	0.2686 7.1964	0.2136 9.3633	0.1849 11.0140	0.1662 12.4009
0.85	1-sided	0.2517 6.5684	0.1996 8.5919	0.1725 10.1338	0.1548 11.4295
	2-sided	0.2758 7.6058	0.2186 9.8752	0.1888 11.6058	0.1694 13.0607
0.90	1-sided	0.2672 7.2003	0.2114 9.3742	0.1823 11.0323	0.1634 12.4263
	2-sided	0.2835 8.1385	0.2239 10.5431	0.1930 12.3790	0.1728 13.9232
0.95	1-sided	0.2831 8.1393	0.2233 10.5461	0.1921 12.3844	0.1719 13.9310
	2-sided	0.2924 8.9621	0.2300 11.58[illegible]1	0.1977 13.5816	0.1767 15.2663
0.99	1-sided	0.2993 9.9338	0.2346 12.8102	0.2012 15.0121	0.1795 16.8667
	2-sided	0.3025 10.6057	0.2367 13.6643	0.2027 16.0072	0.1808 17.9810
$1/(1+\sqrt{\beta})$ which is $\lim \hat{\gamma}_0$ for $(1-\alpha)\to 1$		0.3090	0.2403	0.2052	0.1827

[1/] The upper entry in each cell of the body of the table is $\hat{\gamma}_0$ and the lower entry is $\hat{\lambda}$

<u>Numerical examples</u>: Suppose that $p = 3$ and $\sigma_i^2 = 1$ $(0 \leq i \leq 3)$; then $\beta = p = 3$. If one-sided intervals are desired with $d = 0.5$ and $1-\alpha = 0.95$ then from Table 2 we find that $\hat{\gamma}_0 = 0.3475$, $\hat{\lambda} = 5.6993$. Hence $\hat{N} = [\{(5.6993)1/0.5\}^2] = [129.9] = 130$ and $\hat{N}_0 = 46$, $\hat{N}_1 = \hat{N}_2 = \hat{N}_3 = 28$. For the same specification with $\sigma_0^2 = 1/2$, $\sigma_i^2 = 1$ $(1 \leq i \leq 3)$ we have $\beta = 2p = 6$ and hence $\hat{\gamma}_0 = 0.2770$, $\hat{\lambda} = 7.2350$; thus $\hat{N} = 105$, $\hat{N}_0 = 30$, $\hat{N}_1 = \hat{N}_2 = \hat{N}_3 = 25$. These calculations give an indication of the sensitivity of the allocations and sample sizes to rather large changes in σ_0^2 and the ratio σ_i^2/σ_0^2 $(1 \leq i \leq 3)$ when $\sigma_1^2 = \sigma_2^2 = \sigma_3^2$.

If the experimenter is prepared to assume that $\sigma_i^2 = \sigma^2$ $(0 \leq i \leq p)$ where the actual value of σ^2 is <u>unknown</u>, and he believes that $\sigma_L^2 \leq \sigma^2 \leq \sigma_U^2$ where $\sigma_L^2 < \sigma_U^2$ are known, then this information can be used in <u>designing</u> the experiment, e.g., acting as if $\sigma^2 = \sigma_U^2$ leads to a conservative choice of $\hat{N}$. However, <u>after</u> the experiment has been conducted, when the results are being summarized, the common unknown σ^2 should be estimated using the pooled data, the estimate s^2 being based on $\nu = \hat{N} - (p+1)$ d.f. The estimate should then be used with Dunnett's (1955) formulae for joint confidence statements (analogous to (I) and (II) of (2.1) and (2.2), respectively):

(III) A $100(1-\alpha)$ percent joint <u>one-sided</u> confidence statement

$$\{\mu_0 - \mu_i \leq \bar{x}_0 - \bar{x}_i + t_{\nu,p,\rho}^{(\alpha)} s\sqrt{\frac{1}{\hat{N}_0} + \frac{1}{\hat{N}_1}} \quad (1 \leq i \leq p)\} \qquad (4.1)$$

or

(IV) A $100(1-\alpha)$ percent joint <u>two-sided</u> confidence statement

$$\{\bar{x}_0 - \bar{x}_i - t'^{(\alpha)}_{\nu,p,\rho}\, s\sqrt{\frac{1}{\hat{N}_0} + \frac{1}{\hat{N}_1}} \leq \mu_0 - \mu_i$$

$$\leq \bar{x}_0 - \bar{x}_i + t'^{(\alpha)}_{\nu,p,\rho}\, s\sqrt{\frac{1}{\hat{N}_0} + \frac{1}{\hat{N}_1}} \quad (1 \leq i \leq p)\}. \tag{4.2}$$

Here $t^{(\alpha)}_{\nu,p,\rho}$ $(t'^{(\alpha)}_{\nu,p,\rho})$ is the upper α equicoordinate point of the p-variate t-distribution (p-variate $|t|$-distribution) with d.f. ν and equal correlations $\rho = \hat{N}_1/(\hat{N}_0+\hat{N}_1)$; tables of $t^{(\alpha)}_{\nu,p,\rho}$ are given by Krishnaiah and Armitage (1966) while tables of $t'^{(\alpha)}_{\nu,p,\rho}$ are given by Hahn and Hendrickson (1971).

5. LOSS FROM EQUAL ALLOCATION ON ALL $p + 1$ TREATMENTS

It is of some interest to determine how much is lost in terms of increased sample size if the experimenter uses equal allocation on all $p + 1$ treatments, i.e., $N_0 = N_1 = \ldots = N_p = N/(p + 1)$, instead of the optimal allocation. For this purpose we define the relative efficiency, $RE = \hat{N}/N$. Here $\hat{N}$ and N are the total sample sizes required to guarantee the same joint one-sided (two-sided) confidence coefficient $1-\alpha$ using (2.3) (using (2.4)) for given $(\sigma_0^2,\sigma_1^2,\ldots,\sigma_p^2)$ and specified d when the optimal allocation and equal allocation, respectively, are employed. (Note that $RE \leq 1$, and small values of RE indicate large relative savings by optimal allocation over equal allocation.)

We now determine RE for the important special case $\sigma_i^2 = \sigma^2$ $(0 \leq i \leq p)$. (Other cases can be determined analogously.) For <u>one-sided</u> comparisons we have (ignoring the integer restrictions)

$$N = 2(p+1)\{t^{(\alpha)}_{\infty,p,1/2}(\sigma/d)\}^2. \tag{5.1}$$

Thus

$$RE = \frac{1}{2(p+1)}\left(\frac{\hat{\lambda}}{t^{(\alpha)}_{\infty,p,1/2}}\right)^2 \tag{5.2}$$

for one-sided comparisons. Here $t^{(\alpha)}_{\infty,p,1/2}$ is the upper α-point of the distribution of the maximum of p equicorrelated standard normal random variables with common correlation $\rho = 1/2$; the values of $t^{(\alpha)}_{\infty,p,1/2}$ have been tabulated for selected p and $1-\alpha$ by Gupta, Nagel and Panchapakesan (1973).

For two-sided comparisons $t^{(\alpha)}_{\infty,p,1/2}$ in (5.1) and (5.2) is replaced by $t'^{(\alpha)}_{\infty,p,1/2}$ where $t'^{(\alpha)}_{\infty,p,1/2}$ is the upper α-point of the distribution of the maximum of the absolute values of p equicorrelated standard normal random variables with common correlation $\rho = 1/2$; the values of $(t'^{(\alpha)}_{\infty,p,1/2})^2$ have been tabulated for selected p and $1-\alpha$ by Krishnaiah and Armitage (1965). Entries for $\nu = 60$ in the tables of Hahn and Hendrickson (1971) can be used as conservative approximations to $t'^{(\alpha)}_{\infty,p,1/2}$.

Some representative RE - values for one-sided comparisons (similar results would be obtained for two-sided comparisons) are given in the following table.

Values of RE for One-Sided Comparisons

$1-\alpha$	p		
	2	5	10
0.75	0.9986	0.9741	0.9390
0.95	0.9818	0.9101	0.8433
0.99	0.9759	0.8890	0.8121

It can be seen that the relative savings using the optimal allocation increase with p and $1-\alpha$ as would be expected; also, for fixed p and $1-\alpha$ the difference $N - \hat{N}$ is directly proportional to $(\sigma/d)^2$.

APPENDIX

Formulae for optimal allocation, and details of computation

A.1 Formulae for optimal allocation for one-sided comparisons (Reference B1):

Let $\Phi(\cdot)$ and $\phi(\cdot)$ denote the standard normal distribution and density function, respectively, and let $\Phi_k(\cdot \mid \rho)$ denote the equicoordinate k-variate standard normal distribution function with common correlation ρ. Then the $(\hat{\gamma}_0, \hat{\lambda})$ given in Tables 1 through 9 are the unique solutions of the following simultaneous equations (A.1) and (A.2):

$$\int_{-\infty}^{\infty} \Phi^p\left[\left(\frac{x}{\sqrt{\gamma}} + \lambda\right)\left(\frac{1-\gamma}{\beta}\right)^{1/2}\right] d\Phi(x) = 1-\alpha, \tag{A.1}$$

$$\begin{aligned} &[(1-\beta)\gamma^2-2\gamma+1]\tau\Phi_{p-1}[\tau \mid (1-\gamma)/\{2(1-\gamma)+\gamma\beta\}] \\ &\quad - \frac{(p-1)\gamma(1-\gamma)\beta}{2(1-\gamma)+\gamma\beta}\,\phi(\tau)\Phi_{p-2}\left[\tau\left\{\frac{1-\gamma+\gamma\beta}{3(1-\gamma)+\gamma\beta}\right\}^{1/2} \,\Big|\, \frac{1-\gamma}{3(1-\gamma)+\gamma\beta}\right] = 0, \end{aligned} \tag{A.2}$$

where

$$\tau = \lambda\gamma\left\{\frac{\beta(1-\gamma)}{[1-\gamma+\gamma\beta][2(1-\gamma)+\gamma\beta]}\right\}^{1/2}.$$

A.2 Formulae for optimal allocation for two-sided comparisons (Reference B2).

Here the $(\hat{\gamma}_0, \hat{\lambda})$ given in Tables 1 through 9 are the unique solutions of the following simultaneous equations (A.3) and (A.4):

$$\int_{-\infty}^{\infty} \left\{\Phi\left[\left(\frac{x}{\sqrt{\gamma}} + \lambda\right)\left(\frac{1-\gamma}{\beta}\right)^{1/2}\right] - \Phi\left[\left(\frac{x}{\sqrt{\gamma}} - \lambda\right)\left(\frac{1-\gamma}{\beta}\right)^{1/2}\right]\right\}^p d\Phi(x) = 1-\alpha, \tag{A.3}$$

$$\frac{\lambda[(1-\beta)\gamma^2-2\gamma+1]D_1}{(1-\gamma+\gamma\beta)^{1/2}} - \frac{(p-1)[\beta(1-\gamma)]^{1/2}}{[2(1-\gamma)+\gamma\beta]^{1/2}}\{\phi(\tau)D_2-\phi[\tau\{\frac{2(1-\gamma)+\gamma\beta}{\gamma\beta}\}]D_3\} = 0, \tag{A.4}$$

where

$D_1 = \Phi_{p-1}(-\Delta_1\tau_1,\ \tau_1|(1-\gamma)/\{2(1-\gamma)+\gamma\beta\})$,

$D_2 = \Phi_{p-2}(-\Delta_2\tau_2,\ \tau_2|(1-\gamma)/\{3(1-\gamma)+\gamma\beta\})$,

$D_3 = \Phi_{p-2}(-\tau_3,\ \tau_3|(1-\gamma)/\{3(1-\gamma)+\gamma\beta\})$,

$\tau_1 = \tau$, $\tau_2 = \tau[(1-\gamma+\gamma\beta)/\{3(1-\gamma)+\gamma\beta\}]^{1/2}$, $\tau_3 = \Delta_1\tau_2$,

$\Delta_1 = \{2(1-\gamma)+\gamma\beta\}/\gamma\beta$, $\Delta_2 = \{4(1-\gamma)+\gamma\beta\}/\gamma\beta$, and

$\Phi_k(a,b|\rho) = P\{a \leqq Z_i \leqq b\ (1 \leqq i \leqq k)\}$ where the Z_i are standard normal with $\mathrm{corr}\{Z_i,Z_j\} = \rho$ for $i \neq j$, $1 \leqq i, j \leqq k$.

A.3 Details of computation

The IMSL subroutine ZSYSTM was used to solve the pairs of simultaneous equations (A.1), (A.2) and (A.3), (A.4). The stopping criteria used in arriving at the final solutions were the following: (i) the difference between the left hand and the right hand sides of each equation is less than 1×10^{-6} or (ii) in two successive iterations the corresponding trial values of $\hat{\gamma}_0$ and $\hat{\lambda}$ do not differ in the first six significant digits.

To evaluate a quantity of the form $\Phi_k(a.b|\rho)$ (which includes $\Phi_k(b|\rho)$ as a special case for $a = -\infty$) the following iterated integral representation (see equation (2) of Bechhofer and Tamhane (1974)) was used:

$$\Phi_k(a,b|\rho) = \int_{-\infty}^{\infty}\left\{\Phi\left[\frac{x\rho^{1/2}+b}{(1-\rho)^{1/2}}\right]-\Phi\left[\frac{x\rho^{1/2}+a}{(1-\rho)^{1/2}}\right]\right\}^k d\Phi(x).$$

For $p = 2$ the quantity $\Phi_{p-1}(a,b|\rho)$ reduces to $\Phi(b)-\Phi(a)$ and $\Phi_{p-2}(a,b|\rho) = \Phi_0(a,b|\rho) = 1$. Thus the evaluation of the various expressions is particularly simple for $p = 2$.

To evaluate $\Phi(\cdot)$ the formula (26.2.17) given in Abramowitz and Stegun (1964) was used; this formula is accurate to within $\pm 7.5 \times 10^{-8}$. The Romberg quadrature method was used to evaluate the various integrals. All of the calculations were done on a CDC 6600 computer at Northwestern.

The tabulated values of $\hat{\gamma}_0$ are rounded off in the fourth decimal place while the values of $\hat{\lambda}$ are rounded up in the fourth decimal place (to insure a joint confidence coefficient $\geqq 1-\alpha$).

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ABSTRACT

The problem under consideration is that of estimating simultaneously the differences between the means of $p \geq 2$ test treatments and the mean of a control treatment when the population variances of all $p + 1$ treatments are known. Tables are given which permit the experimenter to find the minimal total number of experimental units, and the optimal allocation of these units among the $p + 1$ treatments, in order to make one-sided or two-sided confidence interval estimates of the differences of interest. These intervals achieve a specified joint confidence coefficient $1-\alpha$ alpha for a specified allowance or "yardstick" associated with the common width of the intervals. The computations for these tables are based on the results of Bechhofer (1969) for one-sided comparisons, and Bechhofer and Nocturne (1972) for two-sided comparisons.

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